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REGULAR TOTALLY DOMATICALLY FULL GRAPHS

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The paper studies bipartite undirected graphs and directed graphs which are regular and totally domatically full.

1. Introduction

The domatic number of a graph was defined by Cockayne and Hedetniemi [4] and the total domatic number by the same authors and Dawes [3]; the total domatic number was studied in [1]. These concepts were defined for undirected graphs. The concept of the domatic number was transferred to directed graphs in [5]. Here we shall transfer also the concept of the total domatic number to directed graphs; for this goal we shall first study bipartite undirected graphs. All considered graphs are finite without loops and multiple edges.

Fundamental results concerning the domination in graphs can be found in Berge's book [2].

A dominating (or totally dominating) set in an undirected graph G is a subset D of the vertex set $V(G)$ of G with the property that for each vertex $x \in V(G) - D$ (or $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. A domatic (or total domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or totally dominating respectively) sets in G.

The domatic (or total domatic) number $d(G)$ (or $d_t(G)$) of G is the maximum number of classes of a domatic (or total domatic respectively) partition of G.

The quoted authors have introduced some further related concepts. A graph G is called domatically full, if $d(G) = \mathrm{md}(G) + 1$, where $\mathrm{md}(G)$ is the minimum degree of a vertex in G. (For any graph G the domatic number $d(G) \leq \mathrm{md}(G) + 1$.) A uniquely domatic graph is a graph G in which there exists exactly one domatic partition with $d(G)$ classes. Analogous concepts may be defined for the total domatic number. A graph G is called totally domatically full, if $d_t(G) = \mathrm{md}(G)$. (For any graph $d_t(G) \leq \mathrm{md}(G)$.) A uniquely total domatic graph is a graph G in which there exists exactly one total domatic partition with $d_t(G)$ classes.

A total domatic partition of G with $d_t(G)$ classes will be called maximal.

In the second section of this paper we shall study this concept for a particular case of undirected graphs, namely bipartite undirected graphs. In the third section we shall study them for directed graphs; we shall apply the results of the second section.

2. Bipartite undirected graphs

In this section we shall consider bipartite undirected graphs. The vertex set of a graph G will be denoted by $V(G)$ or shortly by V; the bipartition classes of a bipartite graph will be denoted by V_1 and V_2.

Theorem 1. *Let G be a bipartite undirected graph, let $d_t(G) = k$. Then there exist at least $k!$ maximal total domatic partitions of G.*

Proof. Let $\mathscr{D} = \{D_1, \ldots, D_k\}$ be a maximal total domatic partition of G. For each $i = 1, \ldots, k$ denote $D_i^1 = D_i \cap V_1$, $D_i^2 = D_i \cap V_2$. We have $D_i^1 \neq \emptyset$ for each $i = 1, \ldots, k$; otherwise no vertex of V_2 would be adjacent to a vertex of D_i and D_i would not be a total dominating set of G. Analogously $D_i^2 \neq \emptyset$ for $i = 1, \ldots, k$. Hence $\mathscr{D}^1 = \{D_1^1, \ldots, D_k^1\}$ is a partition of V_1 and $\mathscr{D}^2 = \{D_1^2, \ldots, D_k^2\}$ is a partition of V_2. For each permutation p of the number set $\{1, \ldots, k\}$ we define the partition $\mathscr{D}(p) = \{D_1(p), \ldots, D_k(p)\}$ of $V(G)$ such that $D_i(p) = D_i^1 \cup D_{p(i)}^2$. Evidently $\mathscr{D}(p)$ is a maximal total domatic partition of G for each p and further $\mathscr{D}(p_1) \neq \mathscr{D}(p_2)$ for $p_1 \neq p_2$. As there are $k!$ permutations of $\{1, \ldots, k\}$, the assertion is proved. □

Corollary 1. *No bipartite graph G with $d_t(G) \geq 2$ is uniquely totally domatic.*

We introduce a weaker concept than the uniquely totally domatic graph. A bipartite graph G will be called quasi-uniquely totally domatic, if for each maximal total domatic partition $\mathscr{D}$ the partitions $\mathscr{D}^1$, $\mathscr{D}^2$ are the same. (This is true for all partitions $\mathscr{D}(p)$ from the proof of Theorem 1.) This concept will be useful in the study of directed graphs.

An interesting class of totally domatically full graphs is the class of regular ones. In such a graph G each vertex has degree $d_t(G)$. If $\mathscr{D}$ is a maximal total domatic partition in G, then each vertex of G is adjacent to exactly one vertex from each class of $\mathscr{D}$.

First we prove a theorem concerning undirected graphs which need not be bipartite.

Theorem 2. *Let G be a regular totally domatically full graph and let $\mathscr{D}$ be a maximal total domatic partition of G. Then all classes of $\mathscr{D}$ have the same cardinality.*

Proof. Let D_1, D_2 be two classes of $\mathcal{D}$. Each vertex of D_1 is adjacent to exactly one vertex of D_2 and each vertex of D_2 is adjacent to exactly one vertex of D_1. This yields a one-to-one correspondence between D_1 and D_2 and thus $|D_1| = |D_2|$. As D_1 and D_2 were chosen arbitrarily, the assertion is true. □

This implies an assertion on bipartite graphs.

Theorem 3. *Let G be a regular totally domatically full bipartite graph, let $\mathcal{D}$ be a maximal total domatic partition of G, let $\mathcal{D}^1$, $\mathcal{D}^2$ have the same meaning as in the proof of Theorem* 1. *Then $|V_1| = |V_2|$ and all classes of $\mathcal{D}^1$ and $\mathcal{D}^2$ have the same cardinality.*

Proof. As G is regular and bipartite, we have $|V_1| = |V_2|$. According to Theorem 2 all classes of $\mathcal{D}$ have the same cardinality p. Let $D \in \mathcal{D}$, let $D^1 = D \cap V_1$, $D^2 = D \cap V_2$. Each vertex of D^1 is adjacent to exactly one vertex of D; as G is bipartite, this vertex is in D^2. Similarly each vertex of D^2 is adjacent to exactly one vertex of D^1 and thus there is a one-to-one correspondence between D^1 and D^2. Hence $|D^1| = |D^2| = p/2$. As D was chosen arbitrarily, this holds for all classes of $\mathcal{D}$. □

Now we shall define an auxiliary concept. If G is a graph, then by $H(G)$ we denote the graph with the vertex set $V(H(G)) = V(G)$ in which two vertices are adjacent if and only if they are connected in G by a path of length 2.

The following theorem holds again for undirected graphs in general.

Theorem 4. *Let G be a regular graph of degree k, let $\mathcal{D} = \{D_1, \ldots, D_k\}$ be a partition of $V(G)$. The partition $\mathcal{D}$ is a total domatic partition of G if and only if each D_i for $i = 1, \ldots, k$ is an independent set in $H(G)$.*

Proof. Suppose that each D_i is an independent set in $H(G)$. Let v be a vertex of G, let $D_i \in \mathcal{D}$. If there exist two distinct vertices of D_i which are adjacent to v, then they are adjacent in $H(G)$, which is a contradiction with the independence of D_i. As the degree of v is k, the vertex v is adjacent to exactly one vertex from D_i. As v and D_i were chosen arbitrarily, this implies the assertion. □

Now suppose that $\mathcal{D}$ is a total domatic partition of G. Consider $D_i \in \mathcal{D}$ and suppose that there exist vertices x, y of D_i which are adjacent in $H(G)$. Then there exists a vertex z of G adjacent to both x and y in G. As z has degree k and is adjacent to at least two vertices of D_i, there exists $D_j \in \mathcal{D}$ such that z is adjacent to no vertex of D_j. But then D_j is not a total dominating set in G, which is a contradiction. Hence all classes of $\mathcal{D}$ are independent sets in $H(G)$.

Corollary 2. *Let G be a regular graph of degree k. Let there exist exactly one partition $\mathscr{D}$ of $V(G)$ into k classes, each of which is an independent set in $H(G)$. Then G is uniquely totally domatic.*

If G is bipartite, then in $H(G)$ no vertex of V_1 is connected with any vertex of V_2. Thus $H(G)$ is the disjoint union of its subgraphs $H^1(G)$, $H^2(G)$ induced by V_1 and V_2 respectively.

Corollary 3. *Let G be a bipartite regular graph of degree k and let there exist exactly one partition of V_1 and exactly one partition of V_2 into k classes, each of which is an independent set in $H(G)$. Then G is quasi-uniquely totally domatic.*

We shall study an extremal case, when the number of edges of $H(G)$ is as large as possible, i.e. $H(G)$ is a disjoint union of two complete k-partite graphs. By $\mathscr{U}(k, m)$ we denote the class of k-regular quasi-uniquely totally domatic bipartite graphs G such that $d_t(G) = k$, the intersection of each class of any maximal total domatic partition $\mathscr{D}$ with V_1 and with V_2 has m vertices and two vertices of G belong to the same class of $\mathscr{D}^1$ (or of $\mathscr{D}^2$) if and only if they both belong to V_1 (or to V_2 respectively) and are not connected by a path of length 2 in G.

By PG(k) we denote the finite projective geometry of order k, i.e. in which each point is incident to $k + 1$ lines and each line is incident to $k + 1$ points. The geometry PG(k) has $k^2 + k + 1$ points and also $k^2 + k + 1$ lines.

Theorem 5. *The class $\mathscr{U}(k, k)$ for a positive integer k is non-empty if and only if there exists a finite projective geometry* PG(k).

Proof. Let PG(k) exist. Choose a point p_0 of PG(k) and a line q_0 incident to p_0. The points incident to q_0 and different from p_0 will be $p_1, \ldots, p_k$; the lines incident to p_0 and different from q_0 will be $q_1, \ldots, q_k$. Let V_1 (or V_2) be the set of all points (or lines) of PG(k) which are not incident to q_0 (or p_0, respectively). Let G be the bipartite graph with the bipartition classes V_1, V_2 in which two vertices are adjacent if and only if they are incident in PG(k). The reader may verify himself that $G \in \mathscr{U}(k, k)$. Each class of $\mathscr{D}^1$ is the set of all points incident to one of the lines $q_1, \ldots, q_k$ except p_0, each class of $\mathscr{D}^2$ is the set of all lines incident to one of the points $p_1, \ldots, p_k$ except q_0.

Now suppose that $\mathscr{U}(k, k) \neq \emptyset$ and let $G \in \mathscr{U}(k, k)$. Then we can consider $\mathscr{D}^1$ and $\mathscr{D}^2$. We shall construct PG(k). The set of points of PG(k) will be $V_1 \cup \{p_0, p_1, \ldots, p_k\}$, its set of lines will be $V_2 \cup \{q_0, q_1, \ldots, q_k\}$, where $p_0, p_1, \ldots, p_k$, q_0, $q_1, \ldots, q_k$ are distinct elements not belonging to $V_1 \cup V_2$. Every point from V_1 is incident to all lines from V_2 which are adjacent to it in G and further it is incident to q_i if and only if it is in D_i^1. Every point p_i for $1 \leq i \leq k$ is incident to all lines from D_i^2 and to q_0. The point p_0 is incident to the lines q_0, $q_1, \ldots, q_k$. To any two of the vertices p_0, $p_1, \ldots, p_k$ there exists exactly one line incident with both of them (their joining line), namely q_0. To any two vertices of V_1 belonging to the same class D_i^1 the joining line is q_i, for two vertices of V_1 not belonging to the same class of $\mathcal{D}^1$ it is the line of V_2 adjacent to both of them in G. For the point p_0 and a point $x \in D_i^1 \subset V_1$ the joining line is q_i. For a point p_i, where $1 \leq i \leq k$ and a point $x \in V_1$ it is the line from D_i^2 to which x is adjacent in G. Evidently the joining line is always unique. Analogously we can find intersection points of all pairs of lines. Therefore the geometry thus constructed is PG(k). □

A graph thus constructed from PG(2) is a circuit of length 8. The graph obtained in this way from PG(3) has the following matrix of adjacency between V_1 and V_2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Corollary 4. *Let k be a power of a prime number. Then $\mathcal{U}(k, k) \neq \emptyset$.*

Theorem 6. *The class $\mathcal{U}(k, 1) \neq \emptyset$ for each positive integer k, the class $\mathcal{U}(k, 2) \neq \emptyset$ for each integer $k \geq 2$.*

Proof. For each positive integer k the complete bipartite graph $K_{k,k}$ belongs to $\mathcal{U}(k, 1)$. The circuit of length 8 is in $\mathcal{U}(2, 2)$. Consider $k \geq 3$. Let $V_1 = \{x_1, \ldots, x_k, x_1', \ldots, x_k'\}$, $V_2 = \{y_1, \ldots, y_k, y_1', \ldots, y_k'\}$ be the bipartition classes of a graph G and let the edge set of G consist of edges $x_i y_i'$, $x_i' y_i$ for $i = 1, \ldots, k$ and $x_i y_j$, $x_i' y_j'$ for any pair of different numbers i, j from the numbers $1, \ldots, k$. The graph G is regular of degree k. For each $i = 1, \ldots, k$ the vertices x_i, x_i' have distance greater than 2. If $i \neq j$, then x_i, x_j are both adjacent to y_h for any h different from both i and j; analogously x_i', x_j' are both adjacent to y_h' for such a number h. The vertices x_i, x_j' are adjacent both to y_i'. Thus the family $\mathcal{D}^1 = \{\{x_1, x_1'\}, \ldots, \{x_k, x_k'\}\}$ is a partition of V_1 into classes with the property that two vertices of V_1 belong to the same class if and only if they are not adjacent in $H(G)$. An analogous assertion holds for $\mathcal{D}^2 = \{\{y_1, y_1'\}, \ldots, \{y_k, y_k'\}\}$. Therefore $G \in \mathcal{U}(k, 2)$. □

The following matrix is the matrix of adjacency between V_1 and V_2 of a graph from $\mathcal{U}(3, 2)$:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Theorem 7. *For $k < m$ the class $\mathcal{U}(k, m)$ is empty.*

Proof. Suppose that there exists a graph $G \in \mathcal{U}(k, m)$ for $k < m$. Let D_1^1, D_2^1 be two classes of $\mathcal{D}^1$. In $H(G)$ the subgraph induced by $D_1^1 \cup D_2^1$ is a complete bipartite graph and has m^2 edges; therefore there are at least m^2 paths of length 2 going from D_1^1 into D_2^1 in G. But, on the other hand, each vertex of V_2 is adjacent to exactly one vertex of D_1^1 and to exactly one vertex of D_2^1 in G and thus it is an inner vertex of exactly one path of length 2 connecting a vertex of D_1^1 with a vertex of D_2^1. As $|V_2| = km < m^2$, this is a contradiction. □

Theorem 8. *For each m there exists a quasi-uniquely totally domatic graph G with $d_t(G) = 2$ in which all classes of $\mathcal{D}^1$ and $\mathcal{D}^2$ have cardinality m.*

Proof. This graph is a circuit of length $4m$. □

3. Directed graphs

Directed graphs considered here are finite without loops and pairs of vertices joined by two or more equally directed edges.

Let G be a directed graph. A subset D of the vertex set $V(G)$ of G is called dominating (or totally dominating), if to each vertex $x \in V(G) - D$ (or to each $x \in V(G)$ respectively) there exist vertices y, z of D such that there exist edges from x to y and from z to x.

Outgoing from this concept, we may define the domatic number and the total domatic number (and further related concepts) of a directed graph quite analogously as in the case of an undirected graph. Note that the total domatic number of a directed graph is well defined only for directed graphs without sources and sinks (analogously as for undirected graphs without isolated vertices).

In order to transfer the results on bipartite undirected graphs to directed graphs, we introduce an auxiliary concept.

Let G be a directed graph with the vertex set $V(G)$. Consider two disjoint sets V_1, V_2 of the same cardinality as $V(G)$. Choose a one-to-one mapping f_1 of $V(G)$

onto V_1 and a one-to-one mapping f_2 of $V(G)$ onto V_2. By $B(G)$ we denote the bipartite undirected graph with the bipartition classes V_1, V_2 in which a vertex $f_1(u) \in V_1$ is adjacent to $f_2(v) \in V_2$ if and only if an edge goes from u to v in G.

Theorem 9. *Let G be a directed graph, let D be a subset of $V(G)$. Then D is a dominating set in G if and only if $f_1(D) \cup f_2(D)$ is a dominating set in $B(G)$.*

Theorem 10. *Let G be a directed graph, let D be a subset of $V(G)$. Then D is a totally dominating set in G if and only if $f_1(D) \cup f_2(D)$ is a totally dominating set in $B(G)$.*

Proofs are left to the reader.

Corollary 5. *Let G be a directed graph. Then $d(B(G)) \geq d(G)$ and $d_t(B(G)) \geq d_t(G)$.*

Corollary 6. *Let G be a directed graph, let $B(G)$ be uniquely domatic and $d(B(G)) = d(G)$. Then G is uniquely domatic.*

Corollary 7. *Let G be a directed graph, let $B(G)$ be quasi-uniquely totally domatic and $d_t(B(G)) = d_t(G)$. Then G is uniquely totally domatic.*

This yields two constructions of uniquely domatic and uniquely totally domatic directed graphs.

Construction 1. Let G_0 be an undirected bipartite graph which is uniquely domatic and regular of degree $k = d(G_0) - 1$. Let V_1, V_2 be bipartition classes of G_0, let $\mathscr{D}$ be the (unique) maximal domatic partition of G_0. Choose a one-to-one mapping g of V_1 onto V_2 which maps each class of $\mathscr{D}$ onto itself. Direct all edges of G_0 from V_1 to V_2. Identify x with $g(x)$ for each $x \in V_1$. The obtained graph will be denoted by G.

Construction 2. Let G_0 be an undirected bipartite graph which is quasi-uniquely totally domatic and regular of degree $k = d_t(G_0)$. Let V_1, V_2 be the bipartition classes of G_0, let $\mathscr{D}$ be a maximal total domatic partition of G_0. Choose a one-to-one mapping g of V_1 onto V_2 which maps each class of $\mathscr{D}$ onto itself and each vertex onto a vertex non-adjacent to it. Direct all edges of G_0 from V_1 to V_2. Identify x with $g(x)$ for each $x \in V_1$. The obtained graph will be denoted by G.

Theorem 11. *The graph G obtained by Construction 1 (or Construction 2) is a uniquely domatic (or uniquely totally domatic, respectively) directed graph.*

Proof. Evidently $G_0 \cong H(G)$ and thus the assertion follows from Corollaries 6 and 7. □

Corollary 8. *Let k, m be positive integers such that either $k = m$ and it is a power of a prime number, or $k \geq 2$, $m = 2$. Then there exists a regular totally domatically full directed graph G such that $d_t(G) = k$, the graph G is uniquely totally domatic and each class of the maximal total domatic partition of G has m vertices.*

At the end we shall present some fundamental assertions on total domatic numbers of directed graphs.

Theorem 12. *Let G be a directed graph with n vertices. Then $d_t(G) \leq n/2$.*

Proof. Let D be a totally dominating set in G. As each vertex of D must be adjacent to a vertex of D and there are no loops, $|D| \geq 2$. This implies the assertion. □

Theorem 13. *Let G* be a directed graph with n vertices with the property that any pair of vertices is joined by at most one edge. Then $d_t(G) \leq n/3$.

Proof. Let D be a totally dominating set in G. Each vertex of D is joined by edges with at least two vertices of D; one of these edges comes into it, the other goes out. Thus the subgraph of G induced by D contains a circuit and $|D| \geq 3$. This implies the assertion. □

Theorem 14. *Let k, n be positive integers, $n \geq 3k$. Then there exists a tournament T with n vertices such that $d_t(T) = k$.*

Proof. The vertex set of T will be the union of two disjoint sets X, Y. The set X is the set of vertices $x(i, j)$ for $1 \leq i \leq k$, $1 \leq j \leq 3$. If $n = 3k$, then $Y = \emptyset$; else it is the set of vertices $y(i)$ for $1 \leq i \leq n - 3k$. For each i the edges go from $x(i, 1)$ to $x(i, 2)$, from $x(i, 2)$ to $x(i, 3)$ and from $x(i, 3)$ to $x(i, 1)$. Let $i_1 < i_2$; if $j_1 = j_2$, then an edge goes from $x(i_1, j_1)$ to $x(i_2, j_2)$ else it goes inversely. Further edges go from $y(i_1)$ to $y(i_2)$ for $i_1 < i_2$. Finally, for $i = 1, \ldots, k$ edges go from each vertex of Y to $x(i, 1)$ and from $x(i, 2)$ and $x(i, 3)$ to each vertex of Y. Thus the tournament T is given. If $n = 3k$, then $d_t(T) \leq k$ according to Theorem 13. If $n > 3k$, then there exists the vertex $y(n - 3k) \in Y$ which has the out-degree k and thus also $d_t(T) \leq k$. Denote $D_i = \{x(i, 1), x(i, 2), x(i, 3)\}$ for $i = 1, \ldots, k$. Then $\mathscr{D} = \{D_1, \ldots, D_{k-1}, D_k \cup Y\}$ is evidently a total domatic partition of T and thus $d_t(T) = k$. □

4. Problems

(1) For which numbers k, m, is the class $\mathscr{U}(k, m)$ non-empty?
(2) For which positive integers k does there exist an undirected graph G with k^2

vertices which is regular of degree k and in which there exists a partition $\mathscr{D} = \{D_1, \ldots, D_k\}$ of the vertex set of G into classes of equal cardinalities with the property that two vertices x, y are connected by a path of length 2 in G if and only if they belong to different classes of $\mathscr{D}$?

Such a graph would be a non-bipartite analogy of the graphs from $\mathscr{U}(k, k)$ and would be uniquely totally domatic. It is possible to look also for further results analogous to those from Section 2 for graphs which are not bipartite in general.

References

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